COORDINATE TRANSFORMATIONS

UNIVERSAL TRANSVERSE MERCATOR/GEOGRAPHIC

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PREFACE

I have been requested on numerous occassions to provide algorithms for performing geographic to UTM and vice versa coordinate transformations. Many of these requests have come from surveyors and engineers. While there exists a multitude of software packages that can readily handle these copmutations, it seems that many individuals are acquiring their own micro-computers or sophisticated desk-top calculators and as a result are desiring to generate their own programs for these computations.

In the past I have had to sift through numerous references to complie these formulas each time I recieved a request. I can now appreciate why these individuals called me rather than compiling this information themselves -- there are as many different sets of algorithms to perform these basic coordinate transformations as there are sources in which to find them. Unfortunately, the computations are not trivial and the problem of which algorithms to use can be confusing.

I have found the formulae outlined in reference [1], with some formulation from references [2] and [3] the simplest for the accuracies required in many applications today and these will be presented here. Less complex formulation does exist for these transformations, however, they usually do not provide the accuracy and stability of the algorithms presented here.

I hope this paper will be useful to a broad spectrum of individuals who at one time or another find the need to perform these very popular coordinate transformations.

ABSTRACT

A simple to use set of algorithms for transforming geographic coordinates into Universal Transverse Mercator (UTM) coordinates and vice versa is presented. The formulation is based on series expansions in terms of the geodetic quantities η and t and are accurate to 1 mm at ± 3 degrees from the central meridian. Numerical examples showing both final and intermediate results are also included. The alogrithms are outlined for use in the northern hemisphere based on the Clarke 1866 ellipsoid as the reference ellipsoid. However, with very minor modifications these formulas could be used in the southern hemisphere or in conjunction with another reference ellipsoid.

INTRODUCTION

The Universal Transverse Mercator projection and grid were adopted in 1974 by the Ontario Ministry of Natural Resources as the official standard geographical referencing grid for the Province. It is called the "Ontario Geographical Referencing Grid".

The UTM is the ellipsoidal Transverse Mercator (TM) to which specific parameters, such as central meridians, have been applied. Ontario, up to latitude 80° N. is divided into 4 zones each 6 degrees wide in longitude. Thus, Toronto is in grid zone 17 with central meridian 81° W. longitude. Figure 1 illustrates the UTM zones covering Ontario.

Each geographic location in the UTM projection is given x and y coordinates, in metres, usually termed easting and northing respectively. The meridian halfway between the zones' bounding meridians is termed the central meridian (CM) and its scale is reduced to 0.9996 of true scale. This reduction was choosen to minimize scale variation in a given zone. If the geodetic ellipsoid of reference is known, then the UTM zone number and the (X,Y) coordinates (i.e., easting and northing) are sufficient to define any point.

The ellipsoidal Earth is used throughout the UTM projection system, but the reference ellipsoid may be subject to change. At present, for all land under Ontarios jurisdiction, the Clarke 1866 ellipsoid is used for the map projection. However, the forthcomming Redfinition and Readjustment of North American Geodetic Horizontal Control Networks (NAD83), once adopted in Ontario, will change this ellipsoid. Until such official announcement is made, however, the Clarke 1866 ellipsoid is the reference ellipsoid for the UTM projection system in Ontario.

NOTATION AND GEODETIC CONSTANTS

The following notation and geodetic constatnts are used in this paper:

a = semi-major axis of the reference ellipsoid.
b = semi-minor axis of the reference ellipsoid.

$e^2 = (a^2 - b^2)/a^2$ = (eccentricity).

$e'^2 = (a^2 - b^2)/b^2$ = (second eccentricity)

ϕ = geographic (or geodetic) latitude, positive north.

ϕ_1 = latitude corresponding to the meridional arc $M=Y$, i.e., foot-point latitude.

λ = geographic (or geodetic) longitude; considered positive west of Greenwich for purposes of these formulas.

λ_o = longitude of grid zone central meridian.

$\Delta\lambda = \lambda - \lambda_o$

ρ = radius of curvature in the meridian

$= a(1 - e^2)/(1 - e^2\sin^2\phi)^{3/2}$

ν = radius of curvature in the prime vertical

$= a/(1 - e^2\sin^2\phi)^{1/2}$

$\eta^2 = e'^2\cos^2\phi$

$t = \tan\phi$

ϕ_1, ν_1, η_1, t_1, and other subscripted quantities refer to foot-point latitude, ϕ_1

abs(W) = absolute value of the argument, W.

The following is a list of geodetic constants used in conjuction with the Clarke 1866 reference ellipsoid:

Ellipsoid =	Clarke 1866
Associated Geodetic datum =	NAD27 (North American Datum 1927)
a =	6 378 206.4
b =	6 356 583.8
c =	6 399 902.551 59
e^2 =	0.006 768 657 997
e'^2 =	0.006 814 784 946

(figure 1 goes here)

TRANSFORMATION OF GEOGRAPHIC COORDINATES INTO UNIVERSAL TRANSVERSE MERCATOR COORDINATES

Geographic coordinates (ϕ, λ) can be transformed into Transverse Mercator (TM) coordinates (X,Y) by applying the general formulae (CAUTION: the (X,Y) here are NOT UTM coordinates yet):

$$X = A(\Delta\lambda) + B(\Delta\lambda)^3 + C(\Delta\lambda)^5 + D(\Delta\lambda)^7$$

(1)

$$Y = M + F(\Delta\lambda)^2 + G(\Delta\lambda)^4 + H(\Delta\lambda)^6 + I(\Delta\lambda)^8$$

where $\Delta\lambda = \lambda - \lambda_0$ is the difference in longitude from the central meridian λ_0 in radians, A,B,C,D,F,G,H, and I are coefficients given by

$$A = \nu\cos\phi$$

$$
\begin{aligned}
B &= (\nu/6)\cos^3\phi\ (1 - t^2 + \eta^2) \\
C &= (\nu/120)\cos^5\phi \begin{bmatrix} (5 - 18t^2 + t^4) + (14\eta^2 + 13\eta^4 + 4\eta^6) \\ - (58t^2\eta^2 + 64t^2\eta^4 + 24t^2\eta^6) \end{bmatrix} \\
D &= (\nu/5040)\cos^7\phi\ (61 - 479t^2 + 179t^4 - t^6) \\
F &= (\nu/2)\sin\phi\cos\phi \\
G &= (\nu/24)\sin\phi\cos^3\phi\ [(5 - t^2) + (9\eta^2 + 4\eta^4)] \\
H &= (\nu/720)\sin\phi\cos^5\phi \begin{bmatrix} (61 - 58t^2 + t^4) \\ + (270\eta^2 + 445\eta^4 + 324\eta^6 + 88\eta^8) \\ - (330t^2\eta^2 + 680t^2\eta^4 + 600t^2\eta^6 + 192t^2\eta^8) \end{bmatrix} \\
I &= (\nu/40320)\sin\phi\cos^7\phi\ (1385 - 3111t^2 + 543t^4 - t^6)
\end{aligned}
\tag{2}
$$

and M is the length of the meridional arc from the Equator to latitude ϕ, and for the Clarke 1866 ellipsoid is given by

$$
\begin{aligned}
M = 6\ 335\ 034.502\ 24227\ (&1.00510\ 89203\ 88050\,\phi \\
&- 0.00510\ 89203\ 88050\sin\phi\cos\phi \\
&- 0.00002\ 16179\ 26721\sin^3\phi\cos\phi \\
&- 0.00000\ 01138\ 17221\sin^5\phi\cos\phi \\
&- 0.00000\ 00006\ 50041\sin^7\phi\cos\phi \\
&- 0.00000\ 00000\ 03872\sin^9\phi\cos\phi \\
&- 0.00000\ 00000\ 00024\sin^{11}\phi\cos\phi\ )
\end{aligned}
\tag{3}
$$

and where in all the above formulae ϕ, λ, λ_o, and $\Delta\lambda$ are expressed in radians and M in metres.

The expression for $\Delta\lambda$ requires further explanation. If the point to be transformed is WEST of the central meridian the $\Delta\lambda$ value must be made negative before using in equations (1). That is (see figure 2)

$$
\Delta\lambda = \begin{cases} \lambda - \lambda_o\ , & \lambda < \lambda_o \quad \text{(point is EAST of CM)} \\ \lambda_o - \lambda\ , & \lambda > \lambda_o \quad \text{(point is WEST of CM).} \end{cases}
\tag{4}
$$

With TM coordinates (X,Y) computed from equation (1) the UTM coordinates E,N (Easting, Northing) are obtained through the equations (see figure 2)

$$E = \begin{cases} 500\,000 + \mathrm{abs}(k_o\ X)\ , & \lambda < \lambda_o \ \text{(point is EAST of CM)} \\ 500\,000 - \mathrm{abs}(k_o\ X)\ , & \lambda > \lambda_o \ \text{(point is WEST of CM).} \end{cases} \tag{5}$$

$$N = k_o\ Y$$

The constant factor $k_o = 0.9996$ in equations (5) is called the central scale factor and its function is to reduce the scale distortion of the UTM projection system.

TRANSFORMATION OF UNIVERSAL TRANSVERSE MERCATOR COORDINATES INTO GEOGRAPHIC COORDINATES

Since the formulas for this transformation require only the TM X-value coordinate we must first translate and scale our UTM easting, E into the corresponding TM X-value by

$$X = (E - 500\,000)/k_o \tag{6}$$

Now the TM coordinates (X,Y) can be transformed into geographic coordinates (ϕ,λ) by applying the general formulae

$$\begin{aligned} \phi &= \phi_1 + P\,\beta^2 + Q\,\beta^4 + R\,\beta^6 + S\,\beta^8 \\ \Delta\lambda &= \sec\phi_1\ (\beta + T\,\beta^3 + U\,\beta^5 + V\,\beta^7) \end{aligned} \tag{7}$$

where β , P,Q,R,S,T,U, and V are coefficients given by

$$\begin{aligned}
\beta &= (X/\nu_1) \\
\gamma &= t_1 + t_1\eta_1^2 \\
P &= -(\gamma/2) \\
Q &= (\gamma/24)[(5 + 3t_1^2) + (\eta_1^2 - 4\eta_1^4) - 9t_1^2\eta_1^2] \\
R &= -(\gamma/720)\Big[(61 + 90t_1^2 + 45t_1^4) \\
&\quad + (46\eta_1^2 - 3\eta_1^4 + 100\,\eta_1^6 + 88\eta_1^8) \\
&\quad - (252t_1^2\eta_1^2 + 66t_1^2\eta_1^4 - 84t_1^2\eta_1^6 + 192t_1^2\eta_1^8) \\
&\quad - (90t_1^4\eta_1^2 - 225t_1^4\eta_1^4)\Big] \\
S &= (\gamma/40320)(1385 + 3633t_1^2 + 4095t_1^4 + 1575t_1^6)
\end{aligned} \tag{8}$$

$$T = -(1/6)(1 + 2t_1^2 + \eta_1^2)$$

$$U = (1/120)\left[\begin{array}{l}(5 + 28t_1^2 + 24t_1^4) + (6\eta_1^2 - 3\eta_1^4 - 4\eta_1^6) \\ + (8t_1^2\eta_1^2 + 4t_1^2\eta_1^4 + 24t_1^2\eta_1^6)\end{array}\right]$$

$$V = -(1/5040)(61 + 662t_1^2 + 1320t_1^4 + 720t_1^6)$$

and ϕ_1 is the latitude corresponding to meridional arc M=Y. This foot-point latitude meridional is best solved for by successive approximations using the following procedure:

$$\phi_{(1)} = Y/A_o c \quad \text{with } M_{(1)} \text{ computed from (3) using } \phi_{(1)} \text{ as latitude argument,}$$

$$\phi_{(2)} = \phi_{(1)} + (Y - M_{(1)})/A_o c \quad \text{with } M_{(2)} \text{ computed from (3) using } \phi_{(2)} \text{ as latitude argument}$$

$$\phi_{(3)} = \phi_{(2)} + (Y - M_{(2)})/A_o c \quad \text{with } M_{(3)} \text{ computed from (3) using } \phi_{(3)} \text{ as latitude agrgument}$$

$$\vdots \qquad (9)$$

$$\phi_{(n)} = \phi_1 \quad \text{when } M_{(n)} = Y ,$$

where,

$$A_o = 1 + (3/4)e^2 + (45/64)e^4 + (175/256)e^6 + (11025/16384)e^8 + (43659/65536)e^{10} + (693693/1048576)e^{12} \qquad (10)$$

Geographic coordinates (ϕ , λ) can now be obtained using equations (7) where (see figure 2)

$$\lambda = \begin{cases} \lambda_o - \text{abs}(\Delta\lambda) , & x > 500\ 000 \text{ (point is EAST of CM)} \\ \lambda_o + \text{abs}(\Delta\lambda) , & x < 500\ 000 \text{ (point is WEST of CM).} \end{cases} \qquad (11)$$

Equations (7) and (12) yield geographic coordinates (ϕ,λ) in radians accurate to five decimal places of one second.

(figure 2 goes here)

NUMERICAL EXAMPLES

Geographic to UTM

Given: ϕ : 43-10-52.40864
λ : 80-22-56.86602
λ_o : 81-00-00 (UTM zone 17)

Convert (ϕ , λ) and λ_o into radians,

ϕ = 0.753 654 544 701 0224
λ = 1.402 938 636 430 972
λ_o = 1.413 716 694 115 407

Hence, the difference in longitude given by equation (4) is

$\Delta\lambda$ = -0.010 778 057 684 4350

because λ is east of the central meridian.

Now using equations (2),(3), and (1) where the value for

e'^2 can be obtained from the list of geodetic constants given above or computed from the given equation we get

t = 0.938 446 031 811 1880

η = 0.003 623 573 115 4539

ν = 6 388 338.711 516 329

A = 4 658 331.288 717 760

B = 50 753.553 301 7645

C = -112 075.887 307 0132

D = -30 943.459 303 1409

F = 1 593 867.130 667 236

G = 293 232.014 772 5329

H = 13 292.810 740 7245

I = -11 104.962 467 6480

M = 4 782 637.642 233 285

X = -50 207.826 872 6695

Y = 4 782 822.800 200 157

Using equations (5) to compute the UTM coordinates (E,N) the result is

$$E = 550\ 187.744 \quad \text{(because the point is east of the CM)}$$
$$N = 4\ 780\ 909.671$$

UTM to Geographic

Given: E : 430 756.720
N : 4 718 544.799
λ_o : 81-00-00 (UTM zone 17)

Translate and scale the UTM coordinates to obtain TM coordinates (X,Y) using equation (6) and convert the central meridian into radians

$$X = -69\ 270.988\ 395\ 3581$$

$$Y = 4\ 720\ 432.972\ 188\ 876$$

$$\lambda_o = 1.\ 413\ 716\ 694\ 115\ 407$$

To compute the latitude corresponding to meridional arc M=Y

use equations (10),(9) and (3) where e^2 can be obtained from the list of geodetic constants given above or computed from the given equation. Equation (9) represents an iterative or successive approximation procedure for computing the foot-point latitude. The following are the numerical values for the successive approximations for this example. The computed value for meridional arc, M converged to the value for Y after nine iterations. Hence the corresponding value for foot-point latitude is $\phi(9)$. The values after each iteration are:

$$A_0 = 1.005\ 108\ 920\ 378\ 586 \text{ (constant for all iterations)}$$

$$\phi(1) = 0.733\ 829\ 804\ 470\ 0183$$
$$M(1) = 4\ 656\ 460.445\ 817\ 601$$

$$\phi(2) = 0.743\ 774\ 855\ 442\ 0517$$
$$M(2) = 4\ 719\ 753.872\ 357\ 474$$

$$\phi(3) = 0.743\ 880\ 427\ 049\ 4506$$
$$M(3) = 4\ 720\ 425.797\ 458\ 893$$

$$\phi(4) = 0.743\ 881\ 542\ 419\ 7422$$
$$M(4) = 4\ 720\ 432.896\ 391\ 271$$

$$\phi(5) = 0.743\ 881\ 554\ 203\ 0979$$
$$M(5) = 4\ 720\ 432.971\ 388\ 110$$

$$\phi(6) = 0.743\ 881\ 554\ 327\ 5834$$
$$M(5) = 4\ 720\ 432.972\ 180\ 416$$

$$\phi(7) = 0.743\ 881\ 554\ 328\ 8985$$
$$M(7) = 4\ 720\ 432.972\ 188\ 786$$

$$\phi(8) = 0.743\ 881\ 554\ 328\ 9124$$
$$M(8) = 4\ 720\ 432.972\ 188\ 874$$

$$\phi(9) = 0.743\ 881\ 554\ 328\ 9126$$
$$M(9) = 4\ 720\ 432.972\ 188\ 876$$

Since $M(9) = Y$ the foot-point latitude is $\phi(9)$.

Now, using equation (8)

$$t_1 = 0.920\ 232\ 618\ 023\ 3872$$

$$\eta_1 = 0.003\ 689\ 994\ 240\ 3482$$

$$\nu_1 = 6\ 388\ 127.327\ 689\ 574$$

$$\beta = -0.010\ 843\ 708\ 154\ 5166$$

$$\gamma = 0.923\ 628\ 271\ 083\ 6740$$

$$P = -0.461\ 814\ 135\ 541\ 8370$$

$$Q = 0.289\ 249\ 457\ 104\ 6944$$

$$R = -0.216\ 321\ 910\ 514\ 2156$$

$$S = 0.191\ 382\ 158\ 964\ 9200$$

$$T = -0.449\ 557\ 689\ 464\ 7838$$

$$U = 0.383\ 076\ 309\ 631\ 1982$$

$$V = -0.397\ 982\ 026\ 367\ 1559$$

Geographic coordinates (ϕ, λ) can now be obtained using equations (7) and (11). This yields

$$\phi = 0.743\ 827\ 255\ 447\ 8973$$

$$\Delta\lambda = -0.014\ 735\ 609\ 304\ 117\ 785$$

and

$\lambda = 1.428\ 452\ 303\ 419\ 525$ (because the point is west of the CM).

The above values for ϕ and λ are in radians. After conversion to degrees-minutes-seconds we have

$$\phi = 42\text{-}37\text{-}05.38473$$

$$\lambda = 81\text{-}50\text{-}39.43759$$

CONCLUSION

The above algorithms for transformations between geographic and UTM coordinates are relatively simple, straightforward and accurate. Simpler, although less accurate formulation is available, however, and the author would be happy to provide the same upon request. It is hoped, however, that the algorithms given here have been presented in such a way as to both demystify some of the mathematics involved in this computation and to present a clear, methodical procedure for performing them.

References

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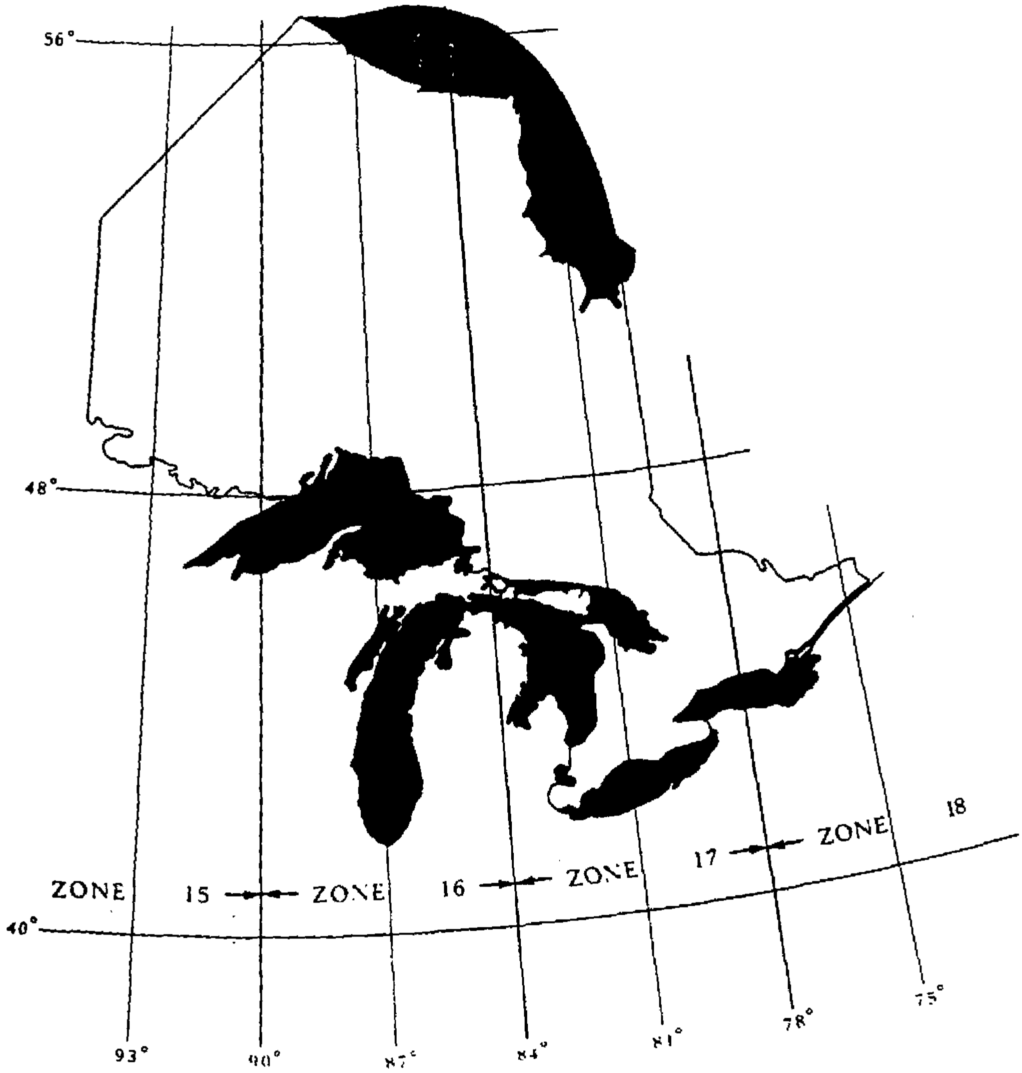

FIGURE 1

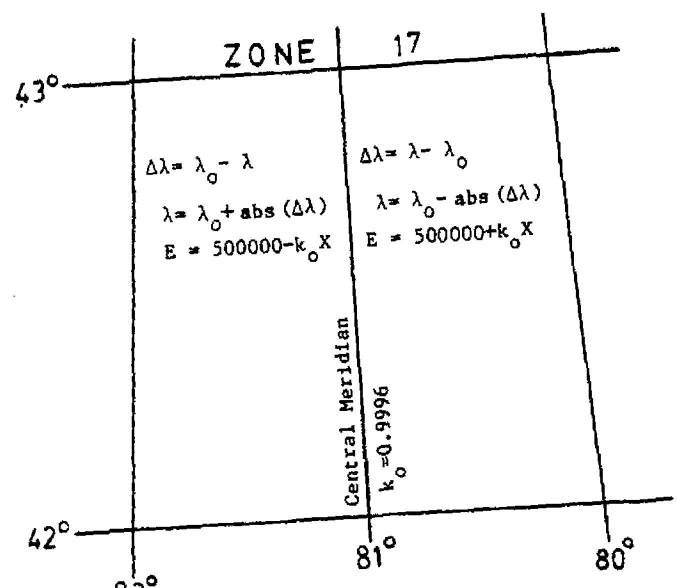

FIGURE 2